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Statistical time moments and an asymptotic formula for the time–energy distribution of slowed-down neutrons

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With 3 figures in the text

SUMMARY

Our starting point ist the exact solution to the problem of slowing-down of neutrons by elastic collisions against free atoms at rest, which has been derived by Waller [1]. From his solution we here derive first the exact time moments and the conditions for their existence. Waller has derived the exact positive moments. Our treatment is somewhat different because we wish to get an analogue treatment of positive and negative moments. Assuming small energies we get, also for the negative moments, the approximate formulae earlier obtained by von Dardel [2] and Walen [3].

Again starting from Waller's solution we derive an asymptotic formula, valid for large values of a variable x, which is the collision number multiplied by the time elapsed since the entering of the neutrons. For small neutron energies we obtain a formula, which can be directly compared with those derived by Marshak [4] and von Dardel [2] by means of the method of moments. We then see that, although these formulae are rather good approximations in the region, where the distribution function is large, they are correct only to the first order if expanded in powers of $1/x$.

Then in a diagram we give the results of numerical calculations for deuterium and carbon moderators based on a formula first derived by Svartholm [5]. As Waller [1] has shown, this formula follows, in a natural way, from the general solution, and it appears to give, for any finite time, the correct asymptotic distribution function for neutrons energies, much smaller than the initial energy. In another diagram we compare von Dardel's [2] distribution function and our asymptotic function with Svartholm's function. Von Dardel's function is here seen to be a fairly good approximation in the experimentally most interesting region.

We also calculate the second non-vanishing term of the expansion obtained for the distribution function by Olsson [6] in the form a power series in time. In order to estimate the rate of convergence we then compare this term with the first non-vanishing term, which has been calculated by Olsson himself.

At last we very briefly discuss time distribution functions for different energies.

Introduction

At the thime $t=0$, Q_0 neutrons with the definite energy $E_0=\frac{1}{2}v_0^2$ ($v_0=$ velocity, mass $=1$) enter an infinite homogeneous and isotropic moderator consisting of free atoms at rest. The neutrons are slowed down by elastic collisions with the atomic nuclei. Capture is neglected. The scattering of the neutrons is assumed to be isotropic in the center-of-mass system.

We introduce the following notations:

M = mass of moderator atom, measured in units of the neutron mass;
$r=(M-1)/(M+1)$;
$\alpha=(M+1)^2/4M$;
$q_M=2\ \ln\ [(M+1)/(M-1)]=\ln\ (1/r^2)$;
v = velocity of a neutron;
$u=\ln\ (v_0^2/v^2)$ = logarithmic energy variable (lethargy);
$l(u)$ = scattering mean free path of a neutron, assumed to depend on u in the following way

$$l(u)/v=\tau\, e^{\varkappa u},$$

where $$\tau=l(0)/v_0$$

and $\varkappa$ is a positive constant. (In particular if $\varkappa=\frac{1}{2}$, we have $l(u)$ const $=l(0)$);
$x=vt/l=(t/\tau)\,e^{-\varkappa u}$;
$\Psi(u,t)$ = the average number of neutron collisions per unit u-interval and unit time interval;
$N(u,t)=(l/v)\,\Psi(u,t)$ = the neutron distribution function.

Under the assumptions made above we then have Waller's [1] solution (Waller, eq. (24))

$$\Psi(u,t)=\frac{Q_0}{\tau}\cdot\frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty} d\eta\, e^{\eta u}\,p(\eta)\int_0^\infty d w\, e^{-w\eta-(t/\tau)e^{-\varkappa w}}\,\frac{1}{2\pi i}\int_{\eta_2-i\infty}^{\eta_2+i\infty} d\eta'\,\frac{e^{w\eta'}}{p(\eta')}, \tag{1}$$

where $$p(\eta)=\frac{\Gamma[(1+\eta+\varkappa)/\varkappa]}{\Gamma[(1+\eta+\varkappa-\alpha)/\varkappa]}\,p_1(\eta) \tag{2a}$$

with $$p_1(\eta)=\prod_{n=1}^{\infty}\left(1+\alpha\,\frac{\exp\,[-q_M(\eta+1+n\varkappa)]}{\eta+1+n\varkappa-\alpha}\right). \tag{2b}$$

$p(\eta)$ can also be defined in the following equivalent way (Waller, eq. (12) and (9)):

$$p(\eta)=e^{-C(\alpha/\varkappa)}\prod_{n=1}^{\infty}G(\eta+n\varkappa)\,e^{\alpha/n\varkappa},\quad C=\text{Euler's constant}, \tag{3a}$$

with $$G(\eta)=1-\frac{\alpha}{\eta+1}\,(1-e^{-q_M(\eta+1)}). \tag{3b}$$

$G(\eta)$ is an analytic function and has one zero at $\eta=0$ and the others in the half plane $\mathrm{Re}\{\eta\}<0$.

The moments

We want to calculate the time moments of the distribution function $N(u, t)$. The sth moment ($s =$ real number) is defined by

$$\langle t^s \rangle = \frac{\int\limits_0^\infty d\,t\,N(u, t)\,t^s}{\int\limits_0^\infty d\,t\,N(u, t)} = \frac{\int\limits_0^\infty d\,t\,\Psi(u, t)\,t^s}{\int\limits_0^\infty d\,t\,\Psi(u, t)}.$$

In order to transform the expression

$$\int\limits_0^\infty d\,t\,\Psi(u, t)\,t^s$$

we use (1) and observe that

$$\int\limits_0^\infty d\,t\,t^s\,e^{-(t/\tau)\varepsilon^{-\varkappa w}} = \Gamma(s+1)\,(\tau\,e^{\varkappa w})^{s+1} \quad \text{if} \quad s \geqslant 0.$$

Thus we get by changing the order of the integrations (if $s < 0$, we have to do a number of integrations by parts in t, before we can change the order of the integrations)

$$\int\limits_0^\infty d\,t\,\Psi(u, t)\,t^s = Q_0\,\tau^s\,\Gamma(s+1)\cdot\frac{1}{2\pi i}\int\limits_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\,e^{\eta u}\,p(\eta)\int\limits_0^\infty d\,w\,e^{-w(\eta-(s+1)\varkappa)}\times$$

$$\times\frac{1}{2\pi i}\int\limits_{\eta_2 - i\infty}^{\eta_2 + i\infty} d\eta'\,\frac{e^{w\eta'}}{p(\eta')} = Q_0\,\tau^s\,\Gamma(s+1)\,\frac{1}{2\pi i}\int\limits_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\,e^{\eta u}\,\frac{p(\eta)}{p(\eta-(s+1)\varkappa)}. \tag{4}$$

If s is a non-negative integer ($s = n \geqslant 0$) we now conclude from (4) with the help of (3a) that

$$\int\limits_0^\infty d\,t\,\Psi(u, t)\,t^n = Q_0\,(\tau\,e^{\varkappa u})^n\,n!\,\frac{1}{2\pi i}\int\limits_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\,e^{\eta u}\prod_{k=0}^{n}\frac{1}{G(\eta + k\varkappa)}. \tag{5}$$

If s is a negative integer ($s = -m < 0$), the expression in the right membrum of (4) does not exist, but we can calculate its limit value (which exists if $u > (m-1)\,q_M$, as we will show later).

$$\frac{Q_0}{\tau^m}\lim_{y\to 0}\Gamma(1-m+y)\,\frac{1}{2\pi i}\int\limits_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\,e^{\eta u}\,\frac{p(\eta)}{p(\eta+(m-1)\varkappa - y\varkappa)}$$

$$= \frac{Q_0}{\tau^m}\,\frac{(-1)^{m-1}}{(m-1)!}\lim_{y\to 0}\frac{1}{y}\cdot\frac{1}{2\pi i}\int\limits_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\,e^{\eta u}\,\frac{p(\eta+(m-1)\varkappa)}{p(\eta+(m-1)\varkappa - \varkappa y)}\prod_{k=1}^{m-1} G(\eta + k\varkappa),$$

since $$\Gamma(1-m+y)=\frac{\Gamma(1+y)}{(1-m+y)\dots y} \quad \text{and} \quad \lim_{y\to 0}\Gamma(1+y)=1.$$

We now put $\varkappa y = z$ and $\eta + m\varkappa = \varrho$ and obtain

$$\int_0^\infty dt\,\Psi(u,t)\,t^{-m} = Q_0\frac{(-1)^{m-1}}{(m-1)!}\varkappa\frac{e^{-m\varkappa u}}{\tau^m}\times$$

$$\times\lim_{z\to 0}\left\{\frac{1}{z}\frac{1}{2\pi i}\int_{\varrho_1-i\infty}^{\varrho_1+i\infty} d\varrho\, e^{\varrho u}\frac{p(\varrho-\varkappa)}{p(\varrho-\varkappa-z)}\prod_{k=1}^{m-1}G(\varrho-k\varkappa)\right\}. \tag{6}$$

Writing $p(\varrho-\varkappa)/p(\varrho-\varkappa-z)$ as $1+[p(\varrho-\varkappa)-p(\varrho-\varkappa-z)]/p(\varrho-\varkappa-z)$ we get

$$\lim_{z\to 0}\frac{1}{z}\cdot\frac{1}{2\pi i}\int_{\varrho_1-i\infty}^{\varrho_1+i\infty} d\varrho\, e^{\varrho u}\frac{p(\varrho-\varkappa)}{p(\varrho-\varkappa-z)}\prod_{k=1}^{m-1}G(\varrho-k\varkappa)$$

$$=\lim_{z\to 0}\left\{\frac{1}{z}\cdot\frac{1}{2\pi i}\int_{\varrho_1-i\infty}^{\varrho_1+i\infty} d\varrho\, e^{\varrho u}\prod_{k=1}^{m-1}G(\varrho-k\varkappa)\right\}+\frac{1}{2\pi i}\int_{\varrho_1-i\infty}^{\varrho_1+i\infty} d\varrho\, e^{\varrho u}\frac{p'(\varrho-\varkappa)}{p(\varrho-\varkappa)}\prod_{k=1}^{m-1}G(\varrho-k\varkappa).$$

If $u>(m-1)q_M$, the integration path of the first integral of the right membrum can be closed to the left, since, according to (3 b), we have

$$e^{\varrho u}\prod_{k=1}^{m-1}G(\varrho-k\varkappa)=O\left(e^{\varrho[u-(m-1)q_M]}\right) \quad \text{if} \quad \operatorname{Re}\{\varrho\}<\varrho_1.$$

The integrand has no poles, and consequently the result of the integration is zero. Thus from (6) we obtain the result

$$\int_0^\infty dt\,\Psi(u,t)\,t^{-m} = Q_0\frac{(-1)^{m-1}}{(m-1)!}\varkappa\frac{e^{-m\varkappa u}}{\tau^m}\cdot\frac{1}{2\pi i}\int_{\varrho_1-i\infty}^{\varrho_1+i\infty} d\varrho\, e^{\varrho u}\frac{p'(\varrho-\varkappa)}{p(\varrho-\varkappa)}\prod_{k=1}^{m-1}G(\varrho-k\varkappa)$$

$$(u>(m-1)q_M) \tag{7}$$

The time moments $$\langle t^n\rangle=\frac{\int_0^\infty dt\,\Psi(u,t)\,t^n}{\int_0^\infty dt\,\Psi(u,t)} \tag{8}$$

can now be calculated from (5) and (7) and we get

$$\boxed{\begin{aligned}
&\langle t^n\rangle = (\tau\, e^{\varkappa u})^n \cdot \frac{1}{N}\cdot\frac{1}{2\pi i}\int_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\, \frac{e^{\eta u}}{G(\eta)} \prod_{k=1}^{n} \frac{k}{G(\eta + k\varkappa)},\\
&\langle t^{-n}\rangle = \varkappa\,(\tau\, e^{\varkappa u})^{-n} \cdot \frac{1}{N}\cdot\frac{1}{2\pi i}\int_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\, e^{\eta u}\, \frac{p'(\eta - \varkappa)}{p(\eta - \varkappa)} \prod_{k=1}^{n-1} \frac{G(\eta - k\varkappa)}{-k},\\
&N = \frac{1}{2\pi i}\int_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\, \frac{e^{\eta u}}{G(\eta)};\quad n > 0.
\end{aligned}} \tag{9}$$

For small neutron energies compared with the initial energy, we have $u \gg q_M$. Then we can calculate the asymptotic value of the integrals of (9), thereby neglecting all residues in the half-plane Re $\{\eta\} < 0$ as being small compared with the residue at the origin. Thus we get

$$\left.\begin{aligned}
\langle t^n\rangle &= (\tau\, e^{\varkappa u})^n \prod_{k=1}^{n} \frac{k}{G(k\varkappa)}\\
\langle t^{-n}\rangle &= \varkappa\, G'(0)\, (\tau\, e^{\varkappa u})^{-n} \prod_{k=1}^{n-1} \frac{G(-k\varkappa)}{-k}
\end{aligned}\right\} \quad (u \gg q_M;\ n > 0). \tag{10}$$

We make the following definitions

$$\varphi(u, x) = \frac{1}{\varkappa x} N(u, x\tau\, e^{\varkappa u}), \tag{11}$$

$$\langle x^n\rangle = \frac{\int_0^\infty d x\, \varphi(u, x)\, x^n}{\int_0^\infty d x\, \varphi(u, x)}. \tag{12}$$

Using the definition of x and (8) we obtain from these equations

$$\langle x^n\rangle = (\tau\, e^{\varkappa u})^{-n} \frac{\langle t^{n-1}\rangle}{\langle t^{-1}\rangle}. \tag{13}$$

For large u, the equations (10) are applicable, and we put those expressions into (13), getting

$$\left.\begin{aligned}
\langle x^n\rangle &= \frac{1}{\varkappa\, G'(0)} \prod_{k=1}^{n-1} \frac{k}{G(k\varkappa)}\\
\langle x^{-n}\rangle &= \prod_{k=1}^{n} \frac{G(-k\varkappa)}{-k}
\end{aligned}\right\} \quad (u \gg q_M;\ n > 0). \tag{14}$$

Since the moments of a function completely determine it, and since the moments of $\varphi(u, x)$ for large u are independent of u, as is seen from (14), we conclude that the limit function $\varphi(x)$ defined by

$$\varphi(x) = \lim_{u\to\infty} \varphi(u, x) = \lim_{u\to\infty} \frac{1}{\varkappa x} N(u, t) \tag{15}$$

exists and is finite (cf. Waller, eq. (48)).

If we have constant mean free path, i.e. $\varkappa = \frac{1}{2}$, then the equations (14) are identical with von Dardel's [2] formulae (von Dardel, (12), (14)), since $\xi = G'(0)$; $1-\lambda_k = G(k/2)$. These formulae have first been obtained by Walen [3].

We now determine under what conditions the negative moments exist by investigating when the integral

$$I_m(u') = \frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty} d\eta\, e^{\eta u'} \frac{p'(\eta-\varkappa)}{p(\eta-\varkappa)} \prod_{k=1}^{m} \frac{G(\eta-k\varkappa)\, e^{\eta q_M}}{-k} \qquad (u' = u - m q_M)$$

appearing in (12) is convergent. Using the logarithmic derivative of (3 a) we obtain

$$I_m(u') = \frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty} d\eta\, e^{\eta u'} \sum_{j=0}^{\infty} \frac{G'(\eta+j\varkappa)}{G(\eta+j\varkappa)} \prod_{k=1}^{m} \frac{G(\eta-k\varkappa)\, e^{\eta q_M}}{-k}$$

and the right membrum converges under the same conditions as

$$\bar{I}_m(u') = \frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty} d\eta\, e^{\eta u'} \sum_{j=0}^{\infty} \frac{1}{\eta+j\varkappa} \prod_{k=1}^{m} \frac{G(\eta-k\varkappa)\, e^{\eta q_M}}{-k} = \sum_{j=0}^{\infty} e^{-j\varkappa u'} a_j,$$

$$a_j = \prod_{k=j+1}^{j+m} \frac{G(-k\varkappa)\, e^{-j\varkappa q_M}}{j-k}.$$

Since, as follows from (3 b),

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = e^{-m\varkappa q_M} \lim_{j\to\infty} \frac{G(-(j+m+1)\varkappa)}{G(-(j+1)\varkappa)} = 1,$$

$\bar{I}_m(u')$, and consequently also $I_m(u')$, converge if $u' > 0$, and diverge if $u' < 0$.

Hence $\langle x^{-m}\rangle$ *exists if* $u' > 0$, *i.e. if* $u > m q_M$ *but does not exist if* $u' < 0$, *i.e. if* $u < m q_M$.

We now see that in order to justify the approximation leading to the second equation (14) we must assume not only large u but also large u', i.e. $u \gg n q_M$.

An approximation for large x (1)

We change the order of the integrations in Waller's formula (1) and write

$$\left.\begin{aligned}\Psi(u,t)&=\frac{Q_0}{\tau}\int_0^{\infty} d\,w\,\theta(w)\,e^{-x e^{\varkappa(u-w)}}\cdot\frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty} d\,\eta\,p(\eta)\,e^{\eta(u-w)},\\ \theta(w)&=\frac{1}{2\pi i}\int_{\eta_2-i\infty}^{\pi_2+i\infty} d\,\eta'\,\frac{e^{w\eta'}}{p(\eta')}.\end{aligned}\right\}\qquad(16)$$

We observe that if $w>u$, we can evaluate the integral

$$\frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty} d\,\eta\,p(\eta)\,e^{\eta(u-w)}$$

by closing the integration path to the right. But the integrand has no poles and consequently the result is zero. This means that we can use u as upper limit in the w-integral in (15). Thus we get

$$\Psi(u,t)=\frac{Q_0}{\tau}\,e^{-x}\int_0^{u} d\,w\,\theta(w)\,e^{-x(e^{\varkappa(u-w)}-1)}\cdot\frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty} d\,\eta\,p(\eta)\,e^{\eta(u-w)}.\qquad(17)$$

For *large values of* x the expression

$$e^{-x(e^{\varkappa(u-w)}-1)}\qquad(18)$$

in (17) can be closely approximated by

$$\exp\{-x[\varkappa(u-w)+\tfrac{1}{2}\varkappa^2(u-w)^2+\tfrac{1}{6}\varkappa^3(u-w)^3]\},\qquad(19)$$

since (19) is a good approximation for (18) if $(u-w)$ is small and for larger $(u-w)$ the expressions (18) and (19) are both so small that we can neglect the contributions to the integral from this region. Then we have for large values of x

$$\Psi(u,t)=\frac{Q_0}{\tau}\,e^{-x}\int_0^{u} d\,w\,\theta(w)\,\exp\,[-x(\varkappa(u-w)+\tfrac{1}{2}\varkappa^2(u-w)^2+\tfrac{1}{6}\varkappa^3(u-w)^3)]\times$$

$$\times\frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty} d\,\eta\,p(\eta)\,e^{\eta(u-w)}.\qquad(20)$$

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(1) The method of treating the triple integral (1) which is used here, and the approximation (19), are due to Professor Waller.

We replace

$$\exp\left[-x\left(\tfrac{1}{2}\varkappa^2(u-w)^2+\tfrac{1}{6}\varkappa^3(u-w)^3\right)\right] \quad \text{by} \quad 1-\tfrac{1}{2}x\varkappa^2(u-w)^2-\tfrac{1}{6}x\varkappa^3(u-w)^3,$$

which is a good approximation for those values of $u-w$ giving the essential contribution to the integral. Now reintroducing ∞ as upper limit in the w-integration, returning to our initial order of integration and inserting the expression (16) for $\theta(w)$, we obtain after some calculation from (20)

$$\Psi(u,t)=\frac{Q_0}{\tau}e^{-x}\left(1-\frac{x}{2}\frac{d^2}{dx^2}-\frac{x}{6}\frac{d^3}{dx^3}\right)\frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty}d\eta\, e^{(\eta-x\varkappa)u}p(\eta)\times$$

$$\times\int_0^\infty dw\, e^{-(\eta-x\varkappa)w}\frac{1}{2\pi i}\int_{\eta_2-i\infty}^{\eta_2+i\infty}d\eta'\,\frac{e^{w\eta'}}{p(\eta')}$$

$$=\frac{Q_0}{\tau}e^{-x}\left(1-\frac{x}{2}\frac{d^2}{dx^2}-\frac{x}{6}\frac{d^3}{dx^3}\right)\frac{1}{2\pi i}\int_{\eta_1-i\infty}^{\eta_1+i\infty}d\eta\, e^{(\eta-\varkappa)u}\frac{p(\eta+(x-1)\varkappa)}{p(\eta-\varkappa)} \tag{21}$$

(cf. Waller, eq. (53a)).

The integral in the right membrum of (21) can be evaluated according to the calculus of residues. If we assume *large* u, we can neglect all residues except the one of the origin, and using (3a) and (3b) we obtain for integral x

$$\boxed{\Psi(u,t)=Q_0\frac{e^{-\varkappa u}}{\tau}e^{-x}\left(1-\frac{x}{2}\frac{d^2}{dx^2}-\frac{x}{6}\frac{d^3}{dx^3}\right)\Gamma_{x-1}} \tag{22}$$

with

$$\boxed{\begin{gathered}\Gamma_{x-1}=\frac{p((x-1)\varkappa)}{G'(0)\,p(0)}=\frac{1}{\xi}\prod_{k=1}^{x-1}\frac{1}{G(k\varkappa)}\\ \xi=G'(0)\end{gathered}} \tag{23}$$

(cf. Waller, eq. (54)). This is our asymptotic expression for large x and u.

Now we are going to introduce an approximation for $p((x-1)\varkappa)$ into (22) in order to get an equation which can be compared with Marshak's [4] and von Dardel's [2] formulae obtained by the methods of moments.

From (2b) we conclude that $p_1((x-1)\varkappa)=1$ up to any order of $1/x$, if x is large enough. (¹) Inserting this in (2a) we get

$$p((x-1)\varkappa)=\Gamma\left(\frac{1}{\varkappa}+x\right)\Big/\Gamma\left(\frac{1-\alpha}{\varkappa}+x\right) \tag{24}$$

(¹) For light moderator nuclei $e^{-q_M}=[(M-1)/(M+1)]^2$ is small and then every factor in the product of (2b) is very close to unity even for rather small x but for heavy nuclei $e^{-q_M}\approx 1$ and we need very large x-values in order to justify the approximation $e^{-q_M(x\varkappa+1)}=0$.

and then by means of (23)

$$\Gamma_{x-1}=\frac{1}{\xi\, p(0)}\cdot\frac{\Gamma(1/\varkappa+x)}{\Gamma[(1-\alpha)/\varkappa+x]}. \tag{25}$$

Using the formula $\Gamma(z)=\sqrt{2\pi}\, z^{z-\frac{1}{2}}\, e^{-z+\frac{1}{2}B_1 1/z+\cdots}$ we can now expand

$$\ln\Gamma(1/\varkappa+x)-\ln\Gamma[(1-\alpha)/\varkappa+x]-(\alpha/\varkappa)\ln x$$

as a power series in $1/x$ and obtain from (22) and (25) to the second order

$$\Psi(u,t)=Q_0\frac{e^{-\varkappa u}}{\tau}\, e^{-x}\,\Gamma_{x-1}\left(1-\frac{2\alpha^2-\alpha}{x}-\frac{\frac{16}{3}\alpha^3-10\alpha^2+\frac{11}{3}\alpha}{x^2}\right). \tag{26}$$

From (25) and (26) we obtain using, the above formula for $\Gamma(z)$, the following equation, which is correct to the first order in $1/x$.

$$\Psi(u,t)\, e^{\varkappa u}\,\tau=\frac{Q_0}{\xi\, p(0)}\, x^{2\alpha}\, e^{-(x+b/x)} \tag{27}$$

with

$$b=4\alpha(\alpha-1)=\frac{(M^2-1)^2}{4M^2}. \tag{28}$$

The right membrum in (27) can now be compared with Marshak's [4] function $F_0(x)$ (Marshak, eq. (50), Table I).

In Table 1 we compare our value of b with those obtained by Marshak and von Dardel [2] (von Dardel, Table 1) for $M=2$, 9 and 15.

Table 1

M	$b=(M^2-1)^2/4M^2$	Marshak's b	von Dardel's b
2	0.56	4.86	0.903
9	19.75	50.15	36.0
12	35.50	—	66.2 (¹)
15	55.75	128.13	105.1

(¹) This value has been obtained from von Dardel's table by means of interpolation.

Marshak's second approximation $F_1(x)$ (Marshak, eq. (54)) cannot be so good, since if $e^x F_1(x)$ is expanded in powers $1/x$, which can be made by successive integrations by parts, this expansion begins with $x^{2\alpha-1}$ instead of $x^{2\alpha}$.

Numerical results for $M=2$ and $M=12$

I have made numerical calculations for $M=2$, $x=0(1)14$ and $M=12$, $x=0(1)14(2)22$ using Svartholm's [5] formula (Svartholm, eq. (6.3)) derived for $\varkappa=\frac{1}{2}$ and large u,

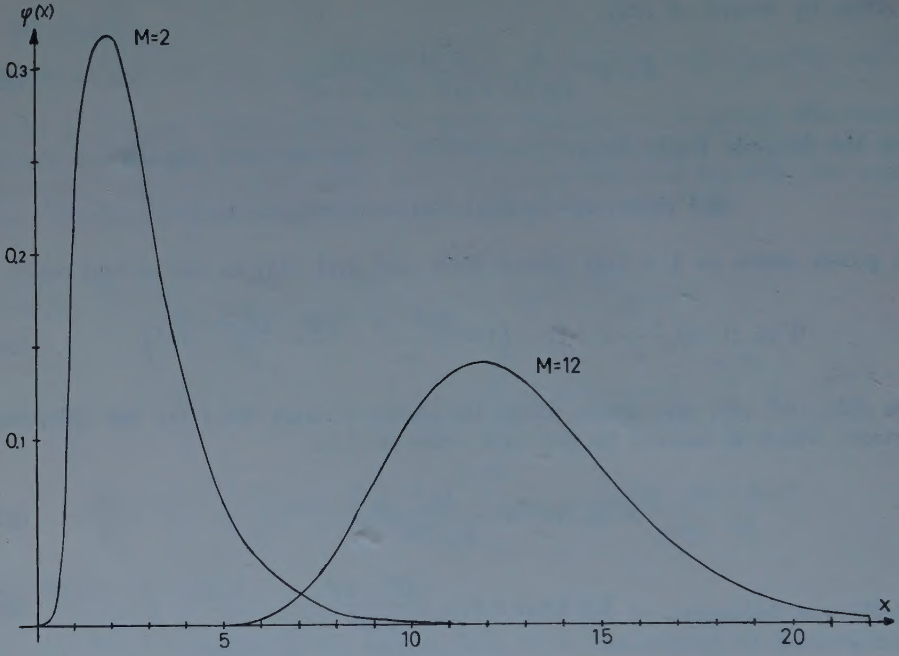

Fig. 1. The distribution functions $\varphi(x)$ for $M=2$ and $M=12$.

$$\varphi(x)=x^{-1}e^{-x}\sum_{n=0}^{\infty}a_n L_n(x), \quad L_n(x)=\text{Laguerre polynomials}$$

(cf. Waller, eq. (42)), taking the first 11 terms for $M=2$ and the first 16 terms for $M=12$. The coefficients were obtained from Svartholm's eq. (4.9), (4.13) with $b_2=2/\xi$. The $L_n(x)$ were taken from tables in an article by Tricomi [7]. The result is indicated by the curves of Fig. 1. Our result for $M=12$ agrees with other calculations based on Svartholm's formula [8].

In Fig. 2 we see that for $M=2$ our asymptotic function obtained from (26) and (11)

$$\varphi(x)=x^{-1}e^{-x}\Gamma_{x-1}\left(1-\frac{45}{32\,x}+\frac{15}{16\,x^2}\right)$$

gives a very good approximation to the right of the maximum. But in the x-range considered here our asymptotic formula for $\varphi(x)$ in the case $M=12$ shows no correlation with the correct formula (cf. note, page 8).

From Fig. 2 we also see that von Dardel's function

$$\varphi(x)=A\,x^{2\alpha-1}\,e^{-x-b/x}$$

(with A such that $\int_0^{\infty}\varphi(x)\,dx=1$) is a very good approximation for $M=2$. This is the case also for $M=12$, but I have not drawn the corresponding curve.

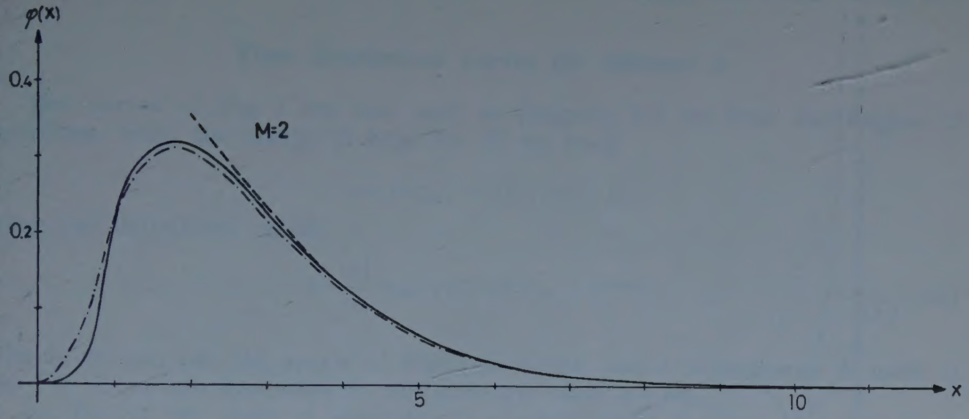

Fig. 2. $M = 2$. $\varphi(x)$. Comparison between Svartholm's exact function (———), von Dardel's function (—·—) and our asymptotic function (— —).

The second non-vanishing term in Olsson's power series

Olsson [6] has derived in the case when $\varkappa = \frac{1}{2}$ an expression for $\Psi(u, t)$ in the form of a power series in t (Olsson, eq. (32)).

$$\Psi(u, t) = \sum_{n=0}^{\infty} K_n t^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\tau}\left(\frac{t}{\tau}\right)^{n+1} \frac{1}{(n+1)!} \cdot \frac{1}{2\pi i} \int_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\, e^{\eta u} \prod_{m=1}^{n+1} G\left(\eta + \frac{m}{2}\right). \quad (29)$$

He has shown that if $m < u/q_M < m+1$, the coefficients of $t^0, t^1, t^2, \ldots t^m$ vanish. He has calculated the first non-vanishing coefficient (Olsson, eq. (35)), for which he obtained

$$K_{m+1} = \frac{(2\alpha)^{m+1}}{2m!\,(m+1)!} \tau^{-(m+2)} \exp\left[-\frac{m^2 + 7m + 6}{4} q_M + \frac{m+3}{2} u'\right] (1 - e^{-\frac{1}{2}u'})^m$$

$$(u' = (m+1) q_M - u). \quad (30)$$

Further he has discussed the structure of the higher coefficients, which are more complicated. According to this discussion the second non-vanishing coefficient is equal to

$$K_{m+2} = (-1)^{m+1} \tau^{-(m+3)} \frac{1}{(m+2)!} \cdot \tfrac{1}{2} M_{m+2}, \quad (31)$$

where M_{m+2} is equal to the sum of residues of the function

$$\sum_{k=0}^{m+1} e^{\frac{1}{2}\zeta u} (2\alpha)^{m+2})(-1) \exp\left(\frac{\zeta + 3 + k}{2} q_M\right) \frac{\exp\left[-\frac{\zeta+3}{2} q_M - \cdots - \frac{\zeta + m + 4}{2} q_M\right]}{(\zeta+3) \cdots (\zeta + m + 4)} +$$

$$+ \sum_{k=0}^{m+1} e^{\frac{1}{2}\zeta u} (2\alpha)^{m+1} \exp\left(\frac{\zeta + 3 + k}{2} q_M\right) (\zeta + 3 + k) \frac{\exp\left[-\frac{\zeta+3}{2} q_M - \cdots - \frac{\zeta + m + 4}{2} q_M\right]}{(\zeta+3) \cdots (\zeta + m + 4)}. \quad (32)$$

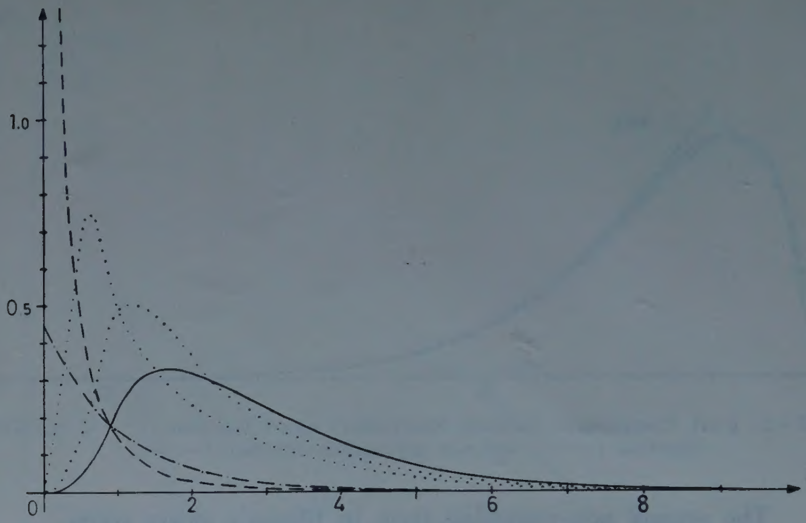

Fig. 3. $M=2$. Comparison between $\varphi(x)$ and $\varphi(u,x)$ for different u.

— — $\frac{1}{5}\varphi_\delta(x)$
—·— $\frac{1}{5}\varphi_1(0,x)$
· · · $\varphi(u,x)$; $1<u/q_M<2$ and $2<u/q_M<3$
——— $\varphi(x)$

After some elementary calculations we get

$$M_{m+2}=\frac{1}{(1-e^{\frac{1}{2}q_M})^2}\frac{(-2\alpha)^{m+1}}{(m+1)!}\exp\left[-\frac{m^2+9m+8}{4}q_M+\frac{m+4}{2}u'\right](1-e^{-\frac{1}{2}u'})^{m+1}\times$$
$$\times\left\{-2\alpha(1-e^{\frac{1}{2}q_M}-e^{\frac{1}{2}(m+2)q_M}+e^{\frac{1}{2}(m+3)q_M})+e^{\frac{1}{2}q_M}-(m+2)e^{\frac{1}{2}(m+2)q_M}+(m+1)\times\right.$$
$$\left.\times\left[e^{\frac{1}{2}(m+3)q_M}-\frac{(1-e^{\frac{1}{2}q_M})(1-e^{\frac{1}{2}(m+2)q_M})}{1-e^{-\frac{1}{2}u'}}\right]\right\}. \tag{33}$$

The ratio $K_{m+2}t^{m+2}/(K_{m+1}t^{m+1})$ is in the case of small energies, i.e. large m, nearly equal to

$$-\frac{1}{1-e^{-\frac{1}{2}q_M}}\cdot\frac{1}{m+2}\cdot\frac{t}{\tau}=-\frac{M+1}{2}\cdot\frac{1}{m+2}\cdot\frac{t}{\tau} \tag{34}$$

and this expression is not greater in magnitude than

$$\frac{M+1}{2}\cdot\frac{q_M}{u}\cdot\frac{t}{\tau}$$

and we conclude that the second non-vanishing term is small compared with the first one, if

$$t\ll\frac{2}{M+1}\cdot\frac{u}{q_M}\cdot\tau. \tag{35}$$

Time distribution curves for different *u*

The curves of Fig. 1 are best used as diagrams for the time distribution of neutrons, when the energy is kept fix. If we keep

$$u = m q_M \quad \text{with} \quad m \geqslant 1,$$

then our definitions entail

$$t = \frac{x\,l}{v} = x \cdot \tau\, e^{\varkappa m q_M} = x \cdot \tau\, r^{-2\varkappa m}. \tag{36}$$

Hence we can use the x-axis of Fig. 1 as t-axis, and t is measured in units of $\tau\, r^{-2\varkappa m}$, if we are interested in the lethargy value $u = m q_M$.

In Fig. 3 we make a *qualitative* (¹) comparison in the case of a deuterium moderator between $\varphi(x)$ and $\varphi(u, x)$, defined in (11), for different u, assuming $\varkappa = \frac{1}{2}$. An exact expression for $\varphi(0, x)$ can be obtained from the basic integro-differential equation (Waller, eq. (1) or Olsson, eq. (1); now $u - q_M < 0$, $Q_0 = 1$)

$$\tau \frac{\partial \Psi(u, t)}{\partial t} = -\Psi(u, t) + \int_0^u d u'\, e^{-(u-u')}\, \Psi(u', t) + \delta(u)\,\delta(t), \tag{37}$$

which has the solution (cf. Olsson, p. 144, first formula)

$$\Psi(u, t) = \frac{1}{\tau} e^{-t/\tau} \left(\delta(u) + \alpha \frac{t}{\tau}\right) \quad \text{if} \quad 0 \leqslant u/q_M < 1. \tag{38}$$

From this we get by means of (11)

$$\varphi(u, x) = \varphi_\delta(x)\,\delta(u) + \varphi_1(u, x), \tag{39}$$

where

$$\varphi_\delta(x) = \frac{2}{x} e^{-x} \tag{40 a}$$

(corresponding to unscattered neutrons), and

$$\varphi_1(0, x) = 2\,\alpha\, e^{-x}. \tag{41 b}$$

To get the form of the curves for *small* x we use Olsson's rule above, according to which we have

$$\varphi(u, x) \approx K_{m+1}\, x^m \quad \text{if} \quad m < u/q_M < m+1.$$

I wish to express my sincere gratitude to Professor Ivar Waller for suggesting the subject of this investigation and for much good advice in the course of the work.

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(¹) Numerical calculations are made only for $\varphi(x)$ and $\varphi(0, x)$. The other $\varphi(u, x)$ (dotted in Fig. 3) are only roughly estimated.

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